ON THE KOBAYASHI HYPERBOLICITY
OF TUBE DOMAINS IN $\mathbb{C}^2$

ALEXANDER ISAEV

ABSTRACT. We construct elementary counterexamples to the criterion for Kobayashi hyperbolicity for a class of tube domains in $\mathbb{C}^2$ proposed by J.-J. Loeb.

1. INTRODUCTION

A connected complex manifold X is said to be *Kobayashi-hyperbolic* if the Kobayashi pseudodistance on X is in fact a distance (see [K] for details). For X endowed with a Riemannian metric, hyperbolicity is equivalent to the following property: for any point $x \in X$ there exist a neighborhood U of x and a constant $M > 0$ such that for all holomorphic maps $f : \Delta \to X$ with $f(0) \in U$ one has $||df(0)|| < M$, where Δ is the unit disk in $\mathbb{C}$. Verification of hyperbolicity may be a difficult task even for very special classes of manifolds.

In this note we focus on *tube domains* in $\mathbb{C}^n$, i.e., domains of the form $T_D := D + i\mathbb{R}^n$, where D is a domain in $\mathbb{R}^n$ called the *base* of T_D. As pointed out in [L] (see also Theorem 13.6.2 in [JP]), for a tube domain $T_D \subset \mathbb{C}^n$ the hyperbolicity property is equivalent to the following condition: for any point $x \in D$ there exist a neighborhood U of x in D and a constant $M > 0$ such that for all harmonic maps $f : \Delta \to D$ with $f(0) \in U$ one has $||df(0)|| < M$.

From now on, we assume that $n = 2$. It is somewhat surprising that so far no easily verifiable criterion for the hyperbolicity of a tube domain has been found even in this situation. By Bochner's theorem, the envelope of holomorphy of T_D coincides with $T_{\widehat{D}}$, where $\widehat{D}$ is the convex hull of D (see, e.g., Section 21 in [V]), and it is natural to investigate hyperbolicity separately in each of the cases: (i) $T_{\widehat{D}} \neq \mathbb{C}^2$ and (ii) $T_{\widehat{D}} = \mathbb{C}^2$. In [HI] we presented several classes of hyperbolic domains in $\mathbb{C}^2$ falling in case (ii). For example, we showed that T_D is hyperbolic if D is a domain bounded by two spirals, where a spiral is a curve defined in polar coordinates in $\mathbb{R}^2$ by the equation $r = g(\varphi)$, with g being an increasing function of φ such that $\lim_{\varphi \to -\infty} g(\varphi) = 0$ and $\lim_{\varphi \to +\infty} g(\varphi) = \infty$. However, there is no comprehensive description of all hyperbolic domains covered by case (ii) (cf. Question 13.6 in [JP]). On the other hand, for domains in $\mathbb{C}^2$ falling in case (i) a hyperbolicity criterion was proposed by J.-J. Loeb in [L], as stated below.

Let $D \subset \mathbb{R}^2$ be a domain with $\widehat{D} \neq \mathbb{R}^2$. Writing coordinates in $\mathbb{C}^2$ as $z_j = x_j + iy_j$, $j = 1, 2$, we may assume without loss of generality that D lies in the half-space $\{x_2 > 0\}$. Then the result of [L] asserts that T_D is hyperbolic if and only if there is no point $(a_1, a_2) \in D$ for which there exists a sequence of real numbers $\{b_k\}$ convergent to a_2 with the property that the segment $[-k, k] \times \{b_k\}$ lies in D for all $k \in \mathbb{N}$. The necessity implication is obvious. Regarding the sufficiency implication, as M. Jarnicky and P. Pflug observed, the argument provided in [L] only yields the following weaker statement (see part (b) of Theorem 13.6.6 in [JP]):

Mathematics Subject Classification: 32Q45.

THEOREM 1.1. *If T_D is not hyperbolic then there exists a point $(a_1, a_2) \in D$ such that for every $k \in \mathbb{N}$ one can find a real-analytic function $\gamma_k(t)$ on $[-k, k]$, with $(t, \gamma_k(t)) \in D$ and $|\gamma_k(t) - a_2| \leq 1/k$ for all t.*

On the other hand, to the best of our knowledge, no counterexample to the sufficiency implication of Loeb's theorem has been found so far (cf. part (a) of Remark 13.6.7 in [JP]). In this note we construct such a counterexample thus clearing the confusion that has existed around Loeb's result for a number of years. Namely, we show:

THEOREM 1.2. *There exists $D \subset \mathbb{R}^2$ lying in the half-space $\{x_2 > 0\}$ such that T_D is not hyperbolic and for $b \in \mathbb{R}$ and $k \in \mathbb{N}$ no segment $[-k, k] \times \{b\}$ is contained in D. Such a domain D can be chosen to have a C^∞-smooth boundary.*

Although the examples provided below are elementary, they are nevertheless surprising as one does expect that obstructions for the hyperbolicity of tube domains should indeed be in some sense "linear" (cf. Loeb's statement). The idea behind the examples inspires a partial converse to Theorem 1.1, which further emphasizes the theme of the "linearity" of obstructions:

THEOREM 1.3. *Let $D \subset \mathbb{R}^2$ be a domain lying in the half-space $\{x_2 > 0\}$ and satisfying the following condition: there exists a point $(a_1, a_2) \in D$ with the property that for every $k \in \mathbb{N}$ one can find an affine function $\gamma_k(t) = c_k t + d_k$ such that $(t, \gamma_k(t)) \in D$ and $|\gamma_k(t) - a_2| \leq 1/k$ for all $t \in [-k, k]$. Then T_D is not hyperbolic.*

However, it is not clear from the proof of Theorem 1.1 given in [JP] that one can always choose the function γ_k appearing there to be affine. Thus, the problem of eliminating the gap between necessary and sufficient conditions for the hyperbolicity of tube domains in $\mathbb{C}^2$ whose envelope of holomorphy is not all of $\mathbb{C}^2$ remains open.

2. The examples and Proof of Theorem 1.3

We start by construct domains satisfying the requirements of Theorem 1.2. First, let

$$D := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < 2\} \setminus \Big(\{-1\} \times [1, 2] \cup \{1\} \times [0, 1] \Big)$$

as shown in Fig. 1 below. Clearly, for $b \in \mathbb{R}$ and $k \in \mathbb{N}$ no segment $[-k, k] \times \{b\}$ is contained in D.

We will now prove that T_D is not hyperbolic. Let $a := (0, 1) \in D$. We will construct a sequence of holomorphic mappings $f_k : \Delta \to T_D$ such that $f_k(0) = a$ and $||df_k(0)|| \to \infty$ as $k \to \infty$. Define

$$f_k : \Delta \to T_D, \quad z \mapsto \left(kz, \frac{1}{k}z + 1 \right).$$

Clearly, $f_k(0) = a$ and

$$df_k(0) = \left(k, \frac{1}{k} \right).$$

Hence, $||df_k(0)|| \to \infty$ as $k \to \infty$, which shows that T_D is not hyperbolic.

The above example can be modified by choosing

$$D := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < 2\} \setminus (S_1 \cup S_2),$$

where

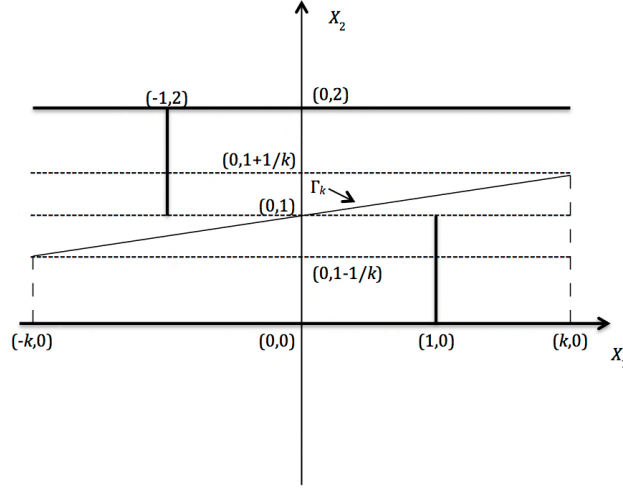$$S_1 \subset \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0, \, 1 \leq x_2 \leq 2\}$$

FIGURE 1.

is a closed region whose boundary contains a curve joining a pair of points on the line $\{x_2 = 2\}$ and passing through a point on the line $\{x_2 = 1\}$, and

$$S_2 \subset \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\, 0 \leq x_2 \leq 1\}$$

is a closed region whose boundary contains a curve joining a pair of points on the line $\{x_2 = 0\}$ and passing through a point on the line $\{x_2 = 1\}$ as shown in Fig. 2 below.
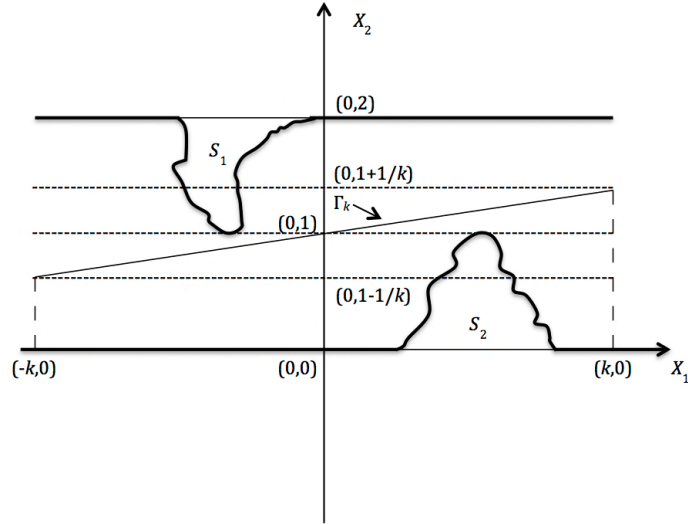

FIGURE 2.

Clearly, S_1, S_2 can be chosen to ensure that ∂D is smooth. Moreover, for any domain of this kind there exists $k_0 \in \mathbb{N}$ such that for $b \in \mathbb{R}$ and $k \geq k_0$ no segment

$[-k, k] \times \{b\}$ is contained in D. It is easy to make a choice of S_1, S_2 so that $k_0 = 1$, which completes the proof of Theorem 1.2. $\qquad\square$

Remark 2.1. Observe that the examples given above satisfy the condition stated in Theorem 1.1 with $a = (0, 1)$ and

$$\gamma_k(t) = \frac{1}{k^2} t + 1.$$

The curve $\Gamma_k(t) := (t, \gamma_k(t))$, with $t \in [-k, k]$, is a line segment as shown in Figs. 1, 2.

The proof of Theorem 1.3 is based on a similar idea. We will construct a sequence of holomorphic mappings $f_k : \Delta \to T_D$ (with k sufficiently large) such that $f_k(0) \to a$ and $||df_k(0)|| \to \infty$ as $k \to \infty$. Define

$$f_k : \Delta \to \mathbb{C}^2, \quad z \mapsto \left(a_1 + \frac{k}{2} z, c_k \left(a_1 + \frac{k}{2} z \right) + d_k \right).$$

Clearly, $f_k(\Delta)$ lies in T_D if k is large enough. Now observe that $c_k \to 0$ and $d_k \to a_2$, which yields $f_k(0) \to a$ as $k \to \infty$. Furthermore,

$$df_k(0) = \frac{1}{2} \left(k, k c_k \right),$$

hence $||df_k(0)|| \to \infty$ as $k \to \infty$. This shows that T_D is not hyperbolic as required. $\qquad\square$

References

[HI]   Huckleberry, A. and Isaev, A., On the Kobayashi hyperbolicity of certain tube domains, *Proc. Amer. Math. Soc.* **141** (2013), 3141–3146.

[JP]   Jarnicki, M. and Pflug, P., *Invariant Distances and Metrics in Complex Analysis*, Second Extended Edition, de Gruyter Expositions in Mathematics, 9, Walter de Gruyter GmbH & Co. KG, Berlin, 2013.

[K]   Kobayashi, S., *Hyperbolic Manifolds and Holomorphic Mappings. An Introduction*, Second Edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[L]   Loeb, J.-J., Applications harmoniques et hyperbolicité de domaines tubes, *Enseign. Math. (2)* **53** (2007), 347–367.

[V]   Vladimirov, V. S., *Methods of the Theory of Functions of Many Complex Variables*, The M.I.T. Press, Cambridge, Mass.-London, 1966.

Mathematical Sciences Institute, Australian National University, Acton, ACT 2601, Australia

E-mail address: alexander.isaev@anu.edu.au